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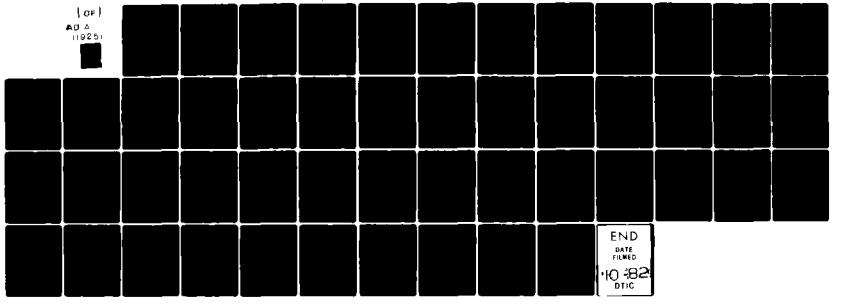

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REGENERATIVE SIMULATION OF HARRIS RECURRENT MARKOV CHAINS

by

Peter W. Glynn

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REGENERATIVE SIMULATION OF HARRIS RECURRENT MARKOV CHAINS

Peter W. Glynn

1. Introduction

In recent years, a substantial amount of research effort has been devoted to development of efficient techniques for statistical analysis of simulation output. When the stochastic system being simulated enjoys regenerative structure, the regenerative property can be exploited to produce a methodology which possesses many attractive properties. Such an approach has proven useful in the steady-state simulation problem (CRANE and IGLEHART (1974), FISHMAN (1973)), quantile estimation (IGLEHART (1976)), and selection of the "best" system (IGLEHART (1977)). The applicability of these procedures has been limited, however, by the need to construct regeneration times for the process under study.

In this chapter, we will extend the regenerative method of simulation output analysis to the class of Harris recurrent Markov chains. These chains are a natural setting for an extension of this kind, since it can be shown that every generalized semi-Markov process (GSMP) corresponds to a certain Markov chain taking values in a complete, separable metric space. As WHITT (1980) has pointed out, GSMPs are a mathematical formulization of the general discrete-event simulation. It is shown in [12] that if the Markov chain corresponding to the GSMP is well-behaved from a simulation standpoint, then the chain must in fact be Harris recurrent. Thus, the class of Harris chains contains the class of well-behaved discrete-event simulations.

In Section 2, we will briefly describe an embedding for Harris chains that provides a weakly regenerative environment (a weakly regenerative process generalizes the concept of regenerative process in the sense that the sequence of tours may be m-dependent). Section 3 develops an algorithm , very similar to that of the regenerative method, for statistical analysis of the steady-state simulation problem. Section 4 concerns a refinement of the algorithm for the case where the embedding is, in fact, regenerative.

For a given Harris chain, there are, in general, many ways of embedding the process in a weakly regenerative environment. In Section 5, we investigate the existence of "optimal" embeddings. Section 6 concerns application of the ideas of Sections 2 through 5 to the "passage time" problem for Harris chains. Recently, there has been substantial interest in passage time problems for networks of queues; see, for example, IGLEHART and SHEDLER (1980). We consider this concept in the Markov chain setting, and prove that one can reduce the passage time problem for Harris chains to the situation covered in Secions 2 through 5. Finally, in Secion 7, we consider a simple vector-valued storage process operating in discrete time, and prove Harris recurrence. This storage process provides examples of chains which can be made regenerative only via an embedding of some kind (such as that developed in Sections 2 and 3). Thus, the techniques considered here do legitimately extend the classical regenerative method.

2. Harris Recurrent Markov Chains

Let E be a complete, separable metric space with $\underline{E}$ its associated Borel sets. Given a probability transition kernel P and a probability μ on $(E,\underline{E})$, one can construct a probability measure P_μ on the product space $\Omega = E \times E \times \cdots$, equipped with product σ-field $\underline{F} = \underline{E} \times \underline{E} \times \cdots$, such that

$$(2.1) \qquad P_\mu\{X_0 \in A_0, X_1 \in A_1, \ldots, X_n \in A_n\}$$
$$= \int_{A_0} \mu(dx_0) \int_{A_1} P(x_0, dx_1) \cdots \int_{A_n} P(x_{n-1}, dx_n)$$

for $A_i \in \underline{E}$, where $X_i(\omega) = \omega_i$ is the projection onto the ith coordinate of Ω. The process $\{X_n\}$ is called the Markov chain associated with kernel P and initial probability μ. We refer the reader to OREY (1971) for notation and additional properties.

The kernel P is said to be a Harris kernel if there exists a probability ϕ on $(E,\underline{E})$, a positive integer m, a non-negative $\underline{E}$-measurable function $\lambda(x)$, and a set $A \in \underline{E}$ such that:

$$(2.2) \qquad \text{i)} \quad P^m(x,\cdot) \geq \lambda(x)\,\phi(\cdot), \quad \text{for all } x \in E,$$
$$\text{ii)} \quad \inf\{\lambda(x): x \in A\} = \lambda > 0,$$
$$\text{iii)} \quad P_\mu\{S_A < \infty\} = 1 \text{ for all probabilities } \mu \text{ on } (E,\underline{E}),$$
$$\text{where } S_A = \inf\{n \geq 0: X_n \in A\}.$$
$$\text{iv)} \quad P_x\{X_{nm} \in A \text{ infinitely often}\} = 1 \text{ for all } x \in A.$$

The set of all pairs (ϕ,λ) satisfying (2.2) for some A will be denoted by $\underline{M}_m$. The measure ϕ will be referred to as a splitting

measure. Finally, any chain for which its kernel is Harris will be called Harris recurrent.

It can be shown that the Harris condition (2.2) is equivalent to requiring existence of a σ-finite measure ν such that $\nu(A) > 0$ implies

$$P_x\{X_n \in A \text{ infinitely often}\} = 1 \tag{2.3}$$

for all $x \in E$ (see Section 4, [12]). All Harris chains possess a unique σ-finite invariant measure π. This measure has the property that $\pi(A) > 0$ if and only if (2.3) holds for every $x \in E$ (see Theorem 2.7, REVUZ (1975)). We will use these equivalent formulations in our analysis of the "passage time" problem in Section 6.

Harris chains are of great importance in simulation. As WHITT (1980) has indicated, a reasonable mathematical formalization of the general discrete-event simulaiton is the class of generalized semi-Markov processes. As is well-known, such processes can be regarded as Markov chains taking values in a complete, separable metric space (see [25] for a discussion). On the other hand, any chain for which the associated steady-state simulation problem is well-posed must necessarily be Harris recurrent (see [12], Proposition 3.8 for definition and results). Hence, the study of well-behaved discrete-event simulations leads naturally to the study of Harris chains.

Any Harris chain can be embedded in a weakly regenerative environment (see Section 4 of [12] for definitions and details). The construction of such an environment proceeds as follows.

Under our topological assumptions on $(E,\underline{E})$, there exist regular conditional distributions $P_x(y,dz)$ such that

$$P_x(B) = \int \left[\int L_B(z,y)\, P_x(y,dz)\right] P^m(x,dy)$$

for all $B \in \underline{E}^m$, the m-fold product σ-field of $\underline{E}(P_x(\cdot)$ is the probability on $(\Omega,\underline{F})$ associated with initial distribution concentrated at x). Assume that $\underline{M}_m$ is non-empty, and let $(\phi,\lambda) \in \underline{M}_m$. Put $E_m = E^m \times \{0,1\}$ and let $\underline{E}_m$ be the corresponding Borel σ-field. For $x \in E$ and $B \in \underline{E}^m$, define the kernel $\bar{P}$ by the formlula

(2.4)
$$\bar{P}(x,\ B \times \{0\})$$
$$= (1 - \lambda(x)) \int \left[\int I_B(v,u)\, P_x(u,dv)\right] Q(x,du)$$

$$\bar{P}(x,\ B \times \{1\})$$
$$= \lambda(x)) \int \left[\int I_B(v,u)\, P_x(u,dv)\right] \phi(du)$$

where $Q(x,\ du)$ is defined by the relation

$$P^m(x,\cdot) = \lambda(x)\, \phi(\cdot) + (1 - \lambda(x))\, Q(x,\cdot) \quad .$$

This kernel defines, for each $x \in E$, a probability $\bar{P}_x$ on $E_m \times E_m \times \cdots$ as follows:

$$(2.5)\quad \bar{P}_x\{Y_1 \in B_1, \ldots, Y_n \in B_n, \delta_1 = i_1, \ldots, \delta_n = i_n\}$$

$$= \int_{B_1} \bar{P}(x, dy_1 \times \{i_1\}) \cdots \int_{B_n} \bar{P}(y_{n-1}(m), dy_n \times \{i_n\}) \quad .$$

Here, (Y_i, δ_i) is the ith coordinate projection on $E_m \times E_m \times \cdots$, where Y_i is the m-vector $(Y_i(1), \ldots, Y_i(m))$. Finally, let $E^* = \cup_{j=0}^{\infty} E^j \times A$, with A as in (2.2), and let $\hat{P}_\mu$ be the probability on $E^* \times E_m \times E_m \times \cdots$ given by

$$(2.6)\quad \hat{P}_\mu\{(Y_0(0), \ldots, Y_0(k)) \in B_0;\ N = k;\ (Y_1, \delta_1, \ldots, Y_n, \delta_n) \in B_1\}$$

$$= \int_{B_0} P_\mu\{(X_0, \ldots, X_k) \in dy_0;\ S_A = k\}$$

$$\times \bar{P}_{y_0(k)}\{(Y_1, \delta_1, \ldots, Y_n, \delta_n) \in B_I\}$$

where $Y_0 = (Y_0(0), \ldots, Y_0(N))$ is the coordinate projection on E^*, and $S_A = \inf\{k \geq 0: X_k \in A\}$. We now define the random variables $\{X_n\}$ by the relation

$$(2.7)\qquad \begin{aligned} X_k &= Y_0(k); && k \leq N \\ X_{N+km+j} &= Y_{k+1}(j); && 1 \leq j \leq m \ . \end{aligned}$$

The process $\{X_n\}$ records the consecutive components of the Y_j's. It is not difficult to show that $\{X_n\}$ has marginal distribution P_μ on $(\Omega, \underline{F})$ and hence (2.7) acts as an embedding of $\{X_n\}$ within the process $Y_0, (Y_1, \delta_1), \ldots$ (see Proposition 4.10, [12]).

The process $Y_0, (Y_1,\delta_1), \ldots$ is weakly regenerative of order 1 with respect to the random times $\{T_j: j \geq 2\}$, where $T_0 = 0$ and

$$T_n = \inf\{k > T_{n-1}: \delta_k = 1\} \quad .$$

Basically, this means that the random "tours"

$$\{(T_{k+1}-T_k, (Y_{T_k}, \delta_{T_k}), \ldots, (Y_{T_{k+1}-1}, \delta_{T_{k+1}-1})); k \geq 2\}$$

are identically distributed and 1-dependent (i.e., tours k and j are independent for $|k-j| > 1$); see Section 4 of [12] for a more precise description of the result. One can therefore regard the probability space constructed above as a weak regenerative embedding of the chain $\{X_n\}$.

Before turning to a simulation implementation of the above embedding, let us recall that we can decompose $P^m(x,\cdot)$ into its ϕ-singular and ϕ-absolutely continuous parts, yielding

$$P^m(x,B) = \int_B h(x,y)\, \phi(dy) + P_s^m(x,B)$$

where $P_s^m(x,\cdot)$ is the ϕ-singular part of the decomposition. Of course, by (2.2)(i), $h(x,y) \geq \lambda(x)$ for ϕ a.e. y. We can now state the simulation implementation of the above weak regenerative embedding.

1. Set $\hat{T}_0 \leftarrow 0$, $k \leftarrow 0$, $j \leftarrow 1$.
2. Generate X_0 with distribution μ.
3. If $X_k \in A$, go to 5.

4. Generate X_{k+1} from the distribution $P(X_k,\cdot)$. Set $k \leftarrow k+1$. Go to 3.

5. Set $Z \leftarrow X_k$, $\ell \leftarrow k$.

6. Generate X_{k+1} from the distribution $P(X_k,\cdot)$. Set $k \leftarrow k+1$.

7. If $k < \ell+m$, go to 6.

8. If $h(Z,X_k) = 0$, go to 5.

9. Generate a random variable U uniform on $[0,1]$.

10. If $\lambda(Z)/h(Z,X_k) \leq U$, go to 5.

11. Set $\hat{T}_j \leftarrow \ell+1$, $j \leftarrow j+1$.

The random times $\{\hat{T}_j\}$ produces by this algorithm are weak regeneration times for the process $\{X_n\}$. It should be pointed out that each random variable in the above algorithm must be generated independently of all other random variables (modulo the fact that the distributions involved are correlated).

The above algorithm incorporates one major modification to the embedding given by (2.3) through (2.7), namely the use of "acceptance-rejection" to calculate those times at which $\{X_n\}$ is distributed according to $\phi(\cdot)$. Let U be a uniform variate on $[0,1]$, and observe that

$$\begin{aligned}
&\bar{P}_x\{(X_1, \ldots, X_m) \in B;\ h(x, X_m) > 0;\ U \leq \lambda(x)/h(x, X_m)\} \\
&\quad = \bar{E}_x\{I_A\ \bar{P}_x\{U \leq \lambda(x)/h(x,X_m) \,|\, X_1, \ldots, X_m\}\} \\
&\quad = \bar{E}_x\{I_A\ \lambda(x)/h(x,X_m)\}
\end{aligned}$$

where

$$\wedge = \{(X_1, \ldots, X_m) \in B;\ h(x, X_m) > 0\}$$

The above expression can be simplified as

$$\bar{E}_x\{\bar{P}_x\{\wedge \mid X_m\}\ \lambda(x)/h(x, X_m)\}$$

$$= \bar{E}_x\{\int I_B(v,X_m)\ P_x(X_m,dv)\ \lambda(x)/h(x, X_m);\ h(x,X_m) > 0\}$$

$$= \lambda(x) \int [\int I_B(v,u)\ P_x(u,dv)]\ \phi(du)\ ,$$

which is, we observe, half of relation (2.3). The other half of (2.3) is equally easy to verify, and thus we see that generation of the additional variable U allows us to avoid explicit generation of random vectors with distribution $P_x(u, dv)$. The term "acceptance-rejection" derives from the interpretation of U as being a variable with indicates when to "accept" a variate as having come from the distribution $\phi(\cdot)$.

3. Confidence Interval Estimation for the Steady-State Simulation Problem

Every Harris recurrent chain $\{X_n\}$ with invariant probability π has the property that

$$(3.1) \qquad \sum_{k=0}^{n} f(X_k)/n \to \pi f \equiv \int f(y)\ \pi(dy)$$

P_x a.s. for all $x \in E$, provided $\pi|f| < \infty$ (see Proposition 3.8 of [12]). Given that $\{X_n\}$ is the output of a simulator, a simulator is often interested in obtaining point estimates and confidence intervals for the parameter πf; this is known as the steady-state simulation problem. We will now show that the weak regenerative embedding of Section 2 can be used to advantage in obtaining a sclution to the above problem.

The strong law (3.1) immediately provides a strongly consistent point estimator for πf. Confidence interval estimation depends on a central limit theorem (CLT) for the summands

$$Y_i(f) = \sum_{k=\hat{T}_i}^{\hat{T}_{i+1}-1} f(X_k)$$

$$\tau_i = Y_i(1) \quad .$$

Assume that $\pi|f| < \infty$, and let $\hat{f}(X_k) = f(X_k) - \pi f$.

(3.2) **PROPOSITION.** **Assuming that** $\sigma^2(Y_2(\hat{f})) < \infty$,

$$n^{-1/2}\left(\sum_{k=\hat{T}_2}^{\hat{T}_n-1} \hat{f}(X_k)\right) \Longrightarrow N(0, \sigma_1^2) \tag{3.3}$$

where $\Rightarrow$ denotes weak convergence, $N(0, \sigma^2)$ is a normal distribution with mean 0 and variance σ^2, and

$$\sigma_1^2 = (\hat{E}_\mu(Y_2(\hat{f}))^2 + 2\hat{E}_\mu(Y_2(\hat{f})\ Y_3(\hat{f}))) \quad .$$

Proof. The sequence $\{Y_i(f);\ i \geq 2\}$ is a sequence of identically distributed 1-dependent random variables, so we may apply Theorem 20.1 of BILLINGSLEY (1968) to obtain (3.3) for $\sigma_1^2 > 0$. On the other hand, if $\sigma_1^2 = 0$, then it is easily computed that the variance of

$$n^{-1/2}\Big(\sum_{k=\hat{T}_2}^{\hat{T}_n - 1} \hat{f}(X_k)\Big) \tag{3.4}$$

converges to 0, and hence (3.4) converges to zero in probability. ∎

This suggests the following simulation algorithm.

1. Simulate $\{X_k\}$ to time $\hat{T}_{n+2}$.
2. Compute $Y_2(f), \tau_2, \ldots, Y_{n+1}(f), \tau_{n+1}$.
3. Compute

$$\hat{r}_n = \sum_{k=2}^{n+1} Y_k(f) / \sum_{k=2}^{n+1} \tau_k$$

$$s_n^2 = \sum_{k=2}^{n+1} Y_k^2(f)/n - 2\hat{r}_n \sum_{k=2}^{n+1} \tau_k Y_k(f)/n + \hat{r}_n^2 \sum_{k=2}^{n} \tau_k^2/n$$

$$c_n = \sum_{k=2}^{n} Y_k(f) Y_{k+1}(f)/n - \hat{r}_n \sum_{k=2}^{n} \tau_k Y_{k+1}(f)/n$$

$$- \hat{r}_n \sum_{k=2}^{n} Y_k(f) \tau_{k+1}/n + \hat{r}_n^2 \sum_{k=2}^{n} \tau_k \tau_{k+1}/n$$

$$v_n = s_n^2 + 2c_n \quad .$$

4. Let $\Phi(x) = P\{(N(0,1) \leq x\}$, put $z_\delta = \Phi^{-1}(1-\delta/2)$, and set

$$L_n = \hat{r}_n - z_\delta v_n^{1/2}/\bar{\tau}_n n^{1/2}$$

$$R_n = \hat{r}_n - z_\delta v_n^{1/2}/\bar{\tau}_n n^{1/2}$$

where

$$\bar{\tau}_n = \sum_{2}^{n+1} \tau_k/n \quad .$$

We claim that the random interval $[L_n, R_n]$ is an approximate $100(1-\delta)\%$ confidence interval for $r(f)$, provided $\sigma_1^2 > 0$. This follows from Proposition 3.2 and the "converging together" lemma (Theorem 4.1, [2]), together with the observation that $E\tau_2 < \infty$ when $\{X_n\}$ possesses an invariant probability (see Proposition 4.15, [12]).

The above simulation algorithm requires that the time horizon of the simulation be random, namely $\hat{T}_{n+2}$ time units. The following algorithm deals with fixed length simulations.

1. Simulate $X_0, X_1, \ldots, X_n$.
2. Let $\ell(n) = \max\{k: \hat{T}_k \leq n\}$, and compute $\hat{r}_{\ell(n)}, v_{\ell(n)}$.
3. Set

$$L_n = \hat{r}_{\ell(n)} - z_\delta \, v_{\ell(n)}^{1/2} / \hat{\tau}_{\ell(n)}^{-1/2} \, n^{1/2}$$

$$R_n = \hat{r}_{\ell(n)} + z_\delta \, v_{\ell(n)}^{1/2} / \hat{\tau}_{\ell(n)}^{-1/2} \, n^{1/2}$$

(if $\ell(n) < 2$, report $L_n = R_n = \sum_0^n f(X_k)/n$). Again, the interval $[L_n, R_n]$ is an approximate $100(1-\delta)\%$ confidence interval for πf; this follows from the asymptotic validity of the first interval, application of the random time change theorem (Theorem 17.1, [2]), and another application of the "converging together" lemma.

4. Regenerative Embeddings for a Special Class of Harris Chains

Let $\{X_n\}$ be a Harris chain such that $\underline{M}_1$ is non-empty. Then, several simplifications of Sections 2 and 3 are possible.

Let $\bar{P}$ be the kernel, corresponding to $(\phi, \lambda) \in \underline{M}_1$, defined in Section 2, and let (X_0, δ_0), (X_1, δ_1), ... be the coordinate projections on $E_1 \times E_1 \times \cdots$. Then, take $\hat{P}_\mu$ as the probability on $E_1 \times E_1 \times \cdots$ defined by

$$\hat{P}_\mu\{X_0 \in B_0, \ldots, X_n \in B_n, \delta_0 = i_0, \ldots, \delta_n = i_n\}$$

$$= \int_{B_0} \hat{\mu}(dx_0 \times \{i_0\}) \int_{B_1} \bar{P}(x_0, dx_1 \times \{i_1\}) \cdots \int_{B_n} \bar{P}(x_{n-1}, dx_n \times \{i_n\})$$

where $B_0, \ldots, B_n \in \underline{E}$ and

$$\hat{\mu}(B_0 \times \{0\}) = \mu(B_0) .$$

It is easily verified that $\{X_n\}$ has marginal distribution P_μ. Furthermore, if we put $T_0 = 0$,

$$T_n = \inf\{k > T_{n-1}: \delta_k = 1\} ,$$

the process $\{X_n\}$ is regenerative with respect to the random times $\{T_j\}$ (i.e., weakly regenerative of order 0). The proof of this result is similar to that of Proposition 4.11 of [12].

The simulation algorithm of Section 2 may be appropriately simplified:

1. Set $T_0 \leftarrow 0$, $k \leftarrow 0$, $j \leftarrow 1$.
2. Generate X_0 with distribution μ.
3. Set $Z \leftarrow X_k$.
4. Generate X_{k+1} from the distribution $P(X_k, \cdot)$. Set $k \leftarrow k+1$.
5. If $h(Z, X_k) = 0$, go to 3.
6. Generate a random variable U uniform on $[0,1]$.
7. If $\lambda(Z)/h(Z,X_k) \leq U$, go to 3.
8. Set $T_j \leftarrow k$, $j \leftarrow j+1$.

Assume for the remainder of this section that the invariant measure π is a probability. To obtain a confidence interval for πf, we consider the sequences

$$Y_i(f) = \sum_{k=T_i}^{T_{i+1}-1} f(X_k)$$

$$\tau_i = Y_i(1) \quad .$$

For a simulation on the random time interval $\{0, 1, \ldots, T_{n+1}\}$, we proceed as follows:

1. Compute $Y_1(f), \tau_1, \ldots, Y_n(f), \tau_n$.
2. Compute

$$\hat{r}_n = \sum_{k=1}^{n} Y_k(f) / \sum_{k=1}^{n} \tau_k$$

$$s_n^2 = \sum_{k=2}^{n+1} Y_k^2(f)/n - 2\hat{r}_n \sum_{k=2}^{n+1} \tau_k Y_k(f)/n + \hat{r}_n^2 \sum_{k=2}^{n} \tau_k^2/n$$

Set

$$L_n = \hat{r}_{\ell(n)} - z_\delta s_n / \bar{\tau}_n \, n^{1/2}$$

$$R_n = \hat{r}_{\ell(n)} + z_\delta s_n / \bar{\tau}_n \, n^{1/2}$$

where

$$\bar{\tau}_n = \sum_{k=1}^{n} \tau_k/n \quad .$$

Then, $[L_n,R_n]$ is an approximate $100(1-\delta)\%$ confidence interval for πf, provided $\pi|f| < \infty$ and $0 < \hat{E}_\mu Y_2(|f|)^2 < \infty$.

For a fixed length simulation, let $\ell(n) = \max\{k: T_k \leq n\}$, and use the interval $[L_n,R_n]$ given by

$$L_n = \hat{r}_{\ell(n)} - z_\delta s_{\ell(n)} / \bar{\tau}_{\ell(n)}^{1/2} \; n^{1/2}$$

$$R_n = \hat{r}_{\ell(n)} + z_\delta s_{\ell(n)} / \bar{\tau}_{\ell(n)}^{1/2} \; n^{1/2} \quad .$$

The proofs of thse CLT's follow from the fact that the constant σ_1^2 of Proposition 3.2 reduces to

$$\sigma_1^2 = \hat{E}_\mu Y_2(\hat{f})^2$$

in the regenerative case.

Given the important role that $\hat{E}_\mu Y_2(\hat{f})^2$ plays in the central limit problem, it is convenient to reduce the question of finiteness of this parameter to one involving parameters defined on the original $\{X_n\}$ process alone.

(4.1) PROPOSITION. Assume that M_1 is non-empty and that (ϕ,λ) and A are as in (2.2), with $m = 1$. Then, $\hat{E}_\mu\{(Y_2(|f|)\}^2 < \infty$, provided that:

i) $\pi|f|^{2r} < \infty$, $r > 1$

ii) $E_{\pi_A}\{R_1 \log R_1\}^{2s} < \infty$, $s > 1$

where $R_1 = \inf\{n \geq 1: X_n \in A\}$, $\pi_A(\cdot) = \pi(\cdot \cap A)/\pi(A)$, and $1/r + 1/s > 1$.

Proof. By condition (2.2)(i) and (ii),

$$\lambda\phi(\cdot) \leq \int P(x,\cdot)\, \pi_A(dx)$$

and hence

$$\hat{E}_\mu\{\sum_{k=T_2}^{T_3-1} |f(X_k)|\}^2 = \hat{E}_\mu\{\hat{E}_{X_{T_2}}\{\sum_{k=0}^{T_1-1} |f(X_k)|\}^2$$

$$= \int \hat{E}_x\{\sum_{k=0}^{T_1-1} |f(X_k)|\}^2\, \phi(dx)$$

$$\leq \iint \hat{E}_y\{\sum_{k=0}^{T_1-1} |f(X_k)|\}^2\, \pi_A(dx)\, P(x,dy)/\lambda$$

$$= \hat{E}_{\pi_A}\{\hat{E}_{X_1}\{\sum_{k=0}^{T_1-1} |f(X_k)|\}^2/\lambda$$

$$= \hat{E}_{\pi_A}\{\sum_{k=1}^{\hat{T}_1-1} |f(X_k)|\}^2/\lambda \leq \hat{E}_{\pi_A}\{\sum_{k=0}^{T_2-1} |f(X_k)|\}^2/\lambda$$

where $\hat{T}_1 = \inf\{k \geq 2: \delta_k = 1\}$. Let $R_0 = 0$, $R_n = \inf\{k > R_{n-1}: X_k \in A\}$. Clearly, the last term above can be bounded by

$$(4.2)\quad \sum_{n=2}^{\infty} \sum_{i=1}^{n-1} \hat{E}_{\pi_A}\{(\sum_{k=0}^{R_{n+1}-1} |f(X_k)|)^2;\ \delta_{R_n+1} = 1,\ \delta_{R_j+1} = \delta_{ij},\ j \le n\}/\lambda$$

$$\le \sum_{n=2}^{\infty} \sum_{i=1}^{n-1} \hat{E}_{\pi_A}\{(n+1) \sum_{\ell=0}^{n-1} V_\ell^2;\ \delta_{R_j+1} = \delta_{ij},\ j < n\}/\lambda$$

where we have used the inequality

$$(\sum_{i=1}^{n} x_i)^2 \le n \sum_{i=1}^{n} x_i^2$$

in the second step and V_ℓ denotes the quantity

$$V_\ell = \sum_{k=R_\ell}^{R_{\ell+1}-1} |f(X_k)| \quad .$$

For the individual term in the above iterated sum, we have, by Holder's inequality, a bound of

$$(4.3)\qquad \sum_{\ell=0}^{n} \hat{E}_{\pi_A}\{V_\ell^{2p}\}^{1/p}\ \hat{E}_{\pi_A}\{\delta_{R_j+1} = \delta_{ij},\ j < n\}^{1/q}/\lambda$$

where $1/p + 1/q = 1$ and $p > 1$. A simple "geometric trial" argument (see Proposition 4.11, [12] for an example) shows that we can bound the δ_{R_j+1} terms by $(1-\lambda)^{(n-2)/q}$. On the other hand, if we use the

fact that π_A is an invariant probability for the chain $\{X_{R_n}\}$ (see [20], p. 32), then we obtain

$$\hat{E}_{\pi_A}\{V_\ell^{2p}\} = E_{\pi_A}\{E_{X_{R_{n-1}}}\{V_0^{2p})\}\} = E_{\pi_A}\{V_0^{2p}\} \quad .$$

Combining these bounds with (4.3), and substituting in (4.2), gives the bound

$$2 \sum_{n=2}^{\infty} n^3 \, E_{\pi_A}\{V_0^{2p}\}^{1/p} \, (1-\lambda)^{(n-1)/q}/\lambda$$

so that $\hat{E}_\phi\{Y_2(|f|)\}^2 < \infty$, provided $E_{\pi_A}\{V_0^{2p}\} < \infty$. Duplicating the reasoning of COGBURN (1970), p. 506, shows that conditions 4.1(i), (ii) suffice to guarantee that $E_{\pi_A} V_0^{2p} < \infty$. ∎

Proposition 4.1 shows that if f is a bounded function, then the central limit theorem holds if $E_{\pi_A} R_1^{2+\delta} < \infty$ for some $\delta > 0$.

The confidence intervals introduced thus far in this section make explicit use of the regenerative structure of the process $\{X_n\}$. Another approach to the confidence interval problem is to treat the sequence $\{f(X_n)\}$ as a stationary process. The expectation is that

$$n^{-1/2}\left(\sum_{k=0}^{n} f(X_k) - n\pi f\right) \Longrightarrow N(0, \sigma_2^2) \tag{4.4}$$

where

$$\sigma_2^2 = E_\pi\{\hat{f}(X_0)\}^2 + 2 \sum_{k=1}^{\infty} E_\pi \hat{f}(X_0)\, \hat{f}(X_k) \quad . \tag{4.5}$$

A confidence interval procedure can then be obtained by using the CLT (4.4), in conjunction with a consistent estimator of σ_2^2; see FISHMAN (1978) for a description of procedures based on this idea.

We can now apply a theorem from regenerative process theory to determine conditions under which statements (4.4) and (4.5) are correct. Suppose that the series defined by (4.5) converges absolutely. Then, (4.4) and (4.5) hold, provided $E_{\pi_A} R_1^{3+\delta} < \infty$ and $E_\pi |f(X)|^{6+\delta} < \infty$, for some $\delta > 0$; see Proposition 7.6 of [11], and use a "geometric trial" argument to prove that $E_{\pi_A} R_1^{3+\delta} < \infty$ implies that that $\hat{E}_\mu \tau_2^3 < \infty$.

5. Optimal Splitting Measures

Our development in Sections 2 through 4 shows that Harris chains can be embedded in a weakly regenerative environment. Furthermore, a special class of Harris chains can, in fact be made regenerative. We have seen that the regenerative structure can be used to advantage in constructing confidence intervals for the steady-state simulation problem.

However, this structure can be useful in analyzing a number of other simulation related questions. Procedures that make heavy use of regenerative structure have been developed for quantile estimation

(IGLEHART (1976)), extreme values (IGLEHART (1977)), sequential stopping rules for simulation (LAVENBERG and SAUER (1977)), and selection of the "best" system (RUBINSTEIN (1980)). Given that a regenerative procedure is adopted for handling a given problem, a simulator often has a choice between two or more sequences of regeneration times for the system under study. The concensus among simulators is that one should use the regenerative sequence which minimizes the expected time between regenerations (e.g. [10]).

In this section, we will therefore consider problems associated with construction of a splitting measure which gives rise to the maximal number of regenerations; such a measure will be called optimal.

Let $\{X_n\}$ be a Harris chain possessing an invariant probability π, *and suppose* $\underline{M}_1$ *is non-empty.* Then, $\{(X_n, \delta_n): n \geq 0\}$ is a regenerative process with $\hat{E}_\mu \tau_2 < \infty$ (Proposition 4.15, [12]), and hence, by the strong law for regenerative processes (Theorem 3.1, [11])

$$I_{\{\delta_k=1\}}/n \to c \qquad \hat{P}_\mu \text{ a.s.} \tag{5.1}$$

where c is a constant yet to be determined. Note that the quantity on the left-hand side of (5.1) is the number of regenerations over the first n time units. Applying bounded convergence to (5.1) shows that c is given by

$$c = \lim_{n \to \infty} \sum_{k=1}^{n} \hat{P}_\mu\{\delta_k = 1\}/n \quad . \tag{5.2}$$

By the Markov property,

$$\hat{P}_\mu\{\delta_k = 1\} = \hat{E}_\mu\{\hat{P}_{X_{k-1}}\{\delta = 1\}\} = \hat{E}_\mu\{\lambda(X_{k-1})\} = E_\mu\{\lambda(X_{k-1})\} \quad .$$

Hence, by applying bounded convergence to (3.1), (5.2) implies that

$$c = \lim_{n \to \infty} \sum_{k=0}^{n-1} E_\mu\, \lambda(X_{k-1})/n$$

$$= \int \lambda(x)\, \pi(dx) \quad .$$

Before proceeding, we point out that if (λ,ϕ) satisfies (2.2)(i) with $m = 1$, and if $\int \lambda(x)\, \pi(dx) > 0$, then there automatically exists A such that (2.2)(ii) through (iv) are satisfied (see Proposition 4.3, [12]), and thus $(\phi,\lambda) \in \underline{M}_1$.

(5.3) THEOREM. There exists a pair $(\phi,\lambda) \in \underline{M}_1$ which maximizes

$$\int \lambda(x)\, \pi(dx)$$

over all pairs $(\phi,\lambda) \in \underline{M}_1$.

Proof. Since $\lambda(x) \leq 1$, it must be that

$$s = \sup\{\int \lambda(x)\, \pi(dx): \text{ there exists } \phi \text{ with } (\phi,\lambda) \in \underline{M}_1\}$$

is less than or equal to 1. Since the problem is trivial if $\underline{M}_1$ is empty, we may assume that there exists a sequence $(\phi_k, \lambda_k) \in \underline{M}_1$ such that

$$\lambda_k \equiv \int \lambda_k(x)\ \pi(dx)$$

is increasing to s. Now, observe that

$$\pi(B) = \int P(x,B)\ \pi(dx) \geq \int \lambda_k(x)\ \phi_k(B)\ \pi(dx) = \lambda_k\ \phi_k(B) \quad .$$

There exists p such that for $n \geq p$, $\lambda_n \geq s/2$, and thus, for $n \geq p$,

$$\phi_n(B) \leq \pi(B)/\lambda_n \leq 2\pi(B)/s \quad . \tag{5.4}$$

We now recall that π is a probability on the Borel sets of a complete, separable metric space, and hence π is a tight measure (Theorem 1.4, [2]). Thus, for each $\varepsilon > 0$, there exists an $\underline{E}$-compact set K_ε such that $\pi(K_\varepsilon) > 1-\varepsilon$. Letting K_ε^c denote the set complement of K_ε, we see, by (5.4) that

$$\phi_n(K_\varepsilon^c) \leq 2\pi(K_\varepsilon^c)/s < 2\varepsilon/s$$

for $n \geq p$, and hence $\phi_n(K_\varepsilon) > 1-2\varepsilon/s$ for $n \geq p$, proving that $\{\phi_n;\ n \geq 1\}$ is a tight family of probability measures. Then, by Prohorov's theorem ([2], p. 35-41), the family $\{\phi_n;\ n \geq 1\}$ is sequentially compact. Hence, there exists a subsequence $\{n_k\}$ such that

(5.5) $$\phi_{n_K} \Longrightarrow \phi$$

where ϕ is a probability on $(E,\underline{E})$. But $(\phi_{n_k}, \lambda_{n_k}) \in \underline{M}_1$, so that

$$P(x,B) \geq \lambda_{n_k}(x)\ \phi_{n_k}(B)$$

for all k and this implies that

(5.6) $$P(x,B) \geq \overline{\lim}\ \lambda_{n_k}(x)\ \phi_{n_k}(B) \geq \overline{\lim}\ \lambda_{n_k}(x)\ \underline{\lim}\ \phi_{n_k}(B) \quad ,$$

Let $\lambda(x)$ be the $\underline{E}$-measurable function $\overline{\lim}\ \lambda_{n_k}(x)$, and observe that (5.6) implies that

(5.7) $$P(x,B) \geq \lambda(x)\ \underline{\lim}\ \phi_{n_k}(B)$$

for $B \in \underline{E}$. In particular, (5.7) holds for all open sets $G \in \underline{E}$, and thus

(5.8) $$P(x,G) \geq \lambda(x)\ \underline{\lim}\ \phi_{n_k}(G) \geq \lambda(x)\ \phi(G) \quad ,$$

the last inequality by the weak convergence relation (5.5) (see [2], p. 12). For a fixed x, let $\nu(\cdot)$ be the signed mesure $P(x,\cdot) - \lambda(x)\phi(\cdot)$. Applying Hahn's decomposition to ν (ROYDEN (1968)) allows one to write

$$\nu = \nu^{+} - \nu^{-}$$

where ν^+, ν^- are mutually singular non-negative finite measures. The measures ν^+ and ν^- are outer regular (see p. 402, BREIMAN (1968)) and therefore, given $B \in \underline{E}$, there exist sequences of open sets G_j^1, G_j^2 such that $B \subset G_j^1$, $B \subset G_j^2$, and

$$\nu^+(B) = \lim \nu^+(G_j^1)$$
$$\nu^-(B) = \lim \nu^-(G_j^2) .$$

Let $G_j = G_j^1 \cap G_j^2$, and observe that $B \subset G_j$, so

$$\nu^+(B) \leq \lim \nu^+(G_j) \leq \lim \nu^+(G_j^1) = \nu^+(B)$$
$$\nu^-(B) \leq \lim \nu^-(G_j) \leq \lim \nu^-(G_j^2) = \nu^-(B) .$$

Hence, since the G_j are open, (5.8) implies that

$$\nu(B) = \lim \nu(G_j) \geq 0$$

and thus ν is a non-negative measure. By the remark preceding Theorem 5.3, this proves that $(\lambda,\phi) \in \underline{M}_1$. On the other hand, using the fact that $\lambda(x) \leq 1$ allows us to apply Fatou's lemma to prove that

$$s = \overline{\lim} \int \lambda_{n_k}(x)\, \pi(dx) \leq \int \overline{\lim} \lambda_{n_k}(x)\, \pi(dx) = \int \lambda(x)\, \pi(dx) ,$$

concluding the proof of the theorem. ∎

Thus, there exists a splitting measure ϕ which is optimal over $\underline{M}_1$. Unfortunately, the above existence proof is non-constructive; furthermore, the measure π is, in general, unknown to a simulator, and hence ϕ can not be calculated prior to initiating the simulation. However, under reasonably general conditions, one can find a sub-optimal splitting measure ϕ which is optimal in a certain "maxi-min" sense. Let

$$\Psi(A) = \{\mu: \mu \text{ is a probability on } (E,\underline{E}),\ \mu(A) = 1\} \quad \text{for } A \in \underline{E}\ ,$$

and consider the maxi-min problem,

$$\max_{(\lambda,\phi) \in \underline{M}_1} \ \min_{\mu \in \Psi(A)} \int \lambda(x)\ \mu(dx)\ . \tag{5.9}$$

Given that π is unknown, this problem amounts to maximizing with respect to the "worst" possible distribution of π over A. By letting μ range over the "point mass" probabilities on A, we see that (5.9) is equivalent to

$$\max_{(\lambda,\phi) \in \underline{M}_1} \ \inf_{x \in A} \lambda(x)\ . \tag{5.10}$$

By (2.2)(ii), it is clear that there exist feasible solutions (λ,ϕ) to (5.10), with positive objective value, for certain subsets A.

So simplify the analysis of (5.10), we re-write it in terms of $\tau(\cdot) = \lambda\phi(\cdot)$ as follows:

(5.11) $$\max_{\tau} \tau(E)$$

subject to $P(x,\cdot) \geq \tau(\cdot)$ for $x \in A$. To retrieve (ϕ,λ), set $\lambda = \tau(E)$, $\phi(\cdot) = \tau(\cdot)/\lambda$.

(5.12) PROPOSITION. If $\pi(A) > 0$, there exists a measure τ which solves (5.11).

Proof. Note that for any feasible τ,

$$\tau(\cdot)\ \pi(A) \leq \int_A P(x,\cdot)\ \pi(dx) \leq \pi(\cdot)$$

and hence $\tau(\cdot)$ is absolutely continuous with respect to $\pi(\cdot)$, with derivative $r(\cdot)$ (say). Now, $\underline{E}$ is the Borel field of a separable metric space, so tht $\underline{E}$ is generated by some countable family of subsets $B_0, B_1, B_n, \ldots$ (DELLACHERIE and MEYER (1978), p. 10-11). Letting $\underline{B}_n$ represent the σ-field generated by $B_0, B_1, \ldots, B_n$, every $y \in E$ belongs to a unique atom $A_n(y) \in \underline{B}_n$, so that

$$P(x,B) = \int_B p_n(x,y)\ \pi(dy) + P_s(x,B)\ , \quad \text{for } B \in \underline{B}_n\ ,$$

where $p_n(x,y) = P(x, A_n(y))/\pi(A_n(y))$ (0/0 interpreted as zero) and $P_s(x,\cdot)$ is the π-singular component of $P(x,\cdot)$. Let $p_n(\cdot)$ be the simple function defined by

$$p_n(\cdot) = \inf\{p_n(x,\cdot): x \in A\}$$

and observe that

$$p_n(\cdot) = \inf_{x \in A} P(x, A_n(\cdot))/\pi(A_n(\cdot)) \geq \tau(A_n(\cdot))/\pi(A_n(\cdot)) = r_n(\cdot)$$

where $r_n(\cdot)$ is the π-derivative of τ on $\underline{B}_n$. Let $p(\cdot) = \underline{\lim}\, p_n(\cdot)$, $p(x,\cdot) = \underline{\lim}\, p_n(x,\cdot)$, $\hat{r}(\cdot) = \underline{\lim}\, r_n(\cdot)$. By a classical differentiation theorem (see DOOB (1953), p. 612), $\hat{r}(\cdot) = r(\cdot)$ π a.e., and $p(x,\cdot)$ is the π-derivative of $P(x,\cdot)$. Letting

$$\tau^*(B) = \int_B p(y)\, \pi(dy) ,$$

it is clear that $\tau^*(E) \geq \tau(E)$ for any feasible τ, since $p(\cdot) \geq \hat{r}(\cdot)$. On the other hand, for any $x \in A$,

$$\int_B p(y)\, \pi(dy) = \int_B \underline{\lim}\, \inf_{x \in A} p_n(x,y)\, \pi(dy) \leq \int_B \underline{\lim}\, p_n(x,y)\, \pi(dy)$$

$$\leq P(x,B)$$

and thus τ^* is feasible. The measure τ^* therefore solves (5.11).∎

The proof of Proposition 5.12 constructs the maxi-min optimal measure τ^*, given the invariant probability π. However, the construction of τ^* can often be given in the absence of explicit knowledge of π. Suppose, for example, that the measures $P(x,\cdot)$ are all equivalent to some common measure η, as x ranges over A (two measures are equivalent if they share the same sets of measures 0). Then, a trivial adaptation of the proof of Proposition 5.12 shows that τ^* is given by

$$\tau^*(B) = \int_B \underline{\lim} \inf_{x \in A} \hat{p}_n(x,y)\, \eta(dy) \tag{5.13}$$

where $\hat{p}_n(x,\cdot)$ is the η-derivative of $P(x,\cdot)$ on $\underline{B}_n$.

(5.14) EXAMPLE. Let $\{U_n;\ n \geq 0\}$ be a sequence of independent, identically distributed real-valued random variables with

$$P\{U_n \in dy\} = \begin{cases} \hat{\alpha} \exp(\alpha y)\, dy\,, & y < 0 \\ \hat{\alpha} \exp(-y) dy\,, & y \geq 0\,, \end{cases}$$

where dy is Lebesgue measure and $\hat{\alpha} = \alpha/(\alpha+1)$. The process $W_0 = x$, $W_{n+1} = \max\{W_n + U_n,\ 0\}$ is a Markov chain corresponding to the waiting time process of the M/M/1 queue (see IGLEHART (1971)). If $A = [0,b]$, then (5.13) shows that the Lebesgue derivative of τ^* is given by

$$r(y) = \begin{cases} \hat{\alpha} \exp(\alpha(y-b)), & 0 \leq y \leq \hat{\alpha} b \\ \hat{\alpha} \exp(-y), & y > \hat{\alpha} b \end{cases}$$

We should point out that in actual simulation applications, one would use the maxi-min optimal measure τ^* as follows. Letting $\underline{A}_n$ represents the atoms of $\underline{B}_n$, put

$$\hat{p}_n(x) = \min_{A_i \in \underline{A}_n} P(x,A_i)/\tau^*(A_i)\,, \tag{5.15}$$

which we note is a decreasing sequence of measurable functions bounded below by 1. If we denote the τ^*-derivative of $P(x,\cdot)$ on $\underline{B}_n$ by $\hat{p}_n(x,\cdot)$, then

$$P(x,B) \geq \int_B \underline{\lim}\, \hat{p}_n(x,y)\, \tau^*(dy) \geq \overline{\lim}\, \hat{p}_n(x)\, \tau^*(B) \geq \tau^*(B) \quad ,$$

so that $(\phi,\lambda) \in \underline{M}_1$ where

$$\lambda(x) = \overline{\lim}\, \hat{p}_n(x)\, \tau^*(E), \quad \phi(\cdot) = \tau^*(\cdot)/\tau^*(E) \quad .$$

The algorithm of Section 4 could then be exploited to obtain a regenerative sequence. Of course, in many applications, $\lambda(\cdot)$ can be calculated directly from the densities $\hat{p}(x,y)$ (the τ^*-derivative of $P(x,\cdot)$) without passing through (5.15), namely via

$$\lambda(x) = \inf_y \hat{p}(x,y)\, \tau^*(E) \quad . \tag{5.16}$$

The construction of $\lambda(x)$ via (5.15) is necessary to deal with the case where one is working with a badly behaved version of $\hat{p}$ (with a bad version, (5.16) might lead to $\lambda(x) \equiv 0$).

6. Passage Times for Harris Chains

In Sections 2 through 5, we have developed simulation methodology for Harris chains, and investigated some related questions. The theory considered so far has pertained exclusively to study of problems associated with simulation output analysis of functions of the form $f(X_n)$, where $\{X_n\}$ is the simulated chain and f is a real-valued $\underline{E}$ measurable function. Recently, however, the "passage time" problem has attracted considerable attention in the literature;

see, for example, IGLEHART and SHEDLER (1980). As we shall see, the quantities of interest in the passage time setting can not be expressed in the form $f(X_n)$, and hence we must develop some additional theory to handle this problem.

To simplify the exposition, we shall consider only a special case of the general passage time problem. However, the structure to be imposed here provides enough flexibility so as to cover the large majority of practical cases of interest. We start by assuming that $\{X_n\}$ is a Harris chain with σ-finite invariant measure π. Given four $\underline{E}$-measurable sets A_1, A_2, B_1, B_2, assume that:

(6.1) i) $P_x\{S_k < \infty\} = 1$ for all x, where $S_0 = 0$, and

$$S_{k+1} = \inf\{n > S_k: (X_{n-1}, X_n) \in A_1 \times A_2\}$$

ii) $P_x\{T_k < \infty\} = 1$ for all x, where $T_0 = S_1$, and

$$T_{k+1} = \inf\{n > T_k: (X_{n-1}, X_n) \in B_1 \times B_2\}$$

iii) $P_x\{S_1 < T_1 \leq S_2 < T_2 \leq S_3 < T_3 \leq \cdots\} = 1$ for all x.

The passage time problem is concerned with obtaining estimators associated with the amount of time required for the chain (X_{n-1}, X_n) to make a "passage" from initial configuration $A_1 \times A_2$ to terminal configuration $B_1 \times B_2$. Condition (6.1)(iii) guarantees that the start times $\{S_j\}$ and terminal times $\{T_j\}$ alternate, so that passage time quantities $R_n(f)$ may be defined via the formula

$$R_n(f) = \sum_{j=S_n}^{T_n-1} f(X_j) ,$$

for f a real-valued $\underline{E}$-measurable function.

(6.2) EXAMPLE. Let $\{\theta_t: t \geq 0\}$ be a semi-Markov process taking values in state space $I = \{0, 1, \ldots, n\}$. Let P be a stochastic matrix which describes the evolution of θ_{T_n+}, where T_n are the successive state transition times of the process $\{\theta_t\}$. Also, let $F_i(\cdot)$ be the distribution of the holding time in state i, or alternatively, let

$$F_i(\cdot) = P\{T_{n+1} - T_n \leq \cdot \,\big|\, X_{T_n+} = i\} .$$

The process $\{\theta_t\}$ corresponds to a Markov chain $\{Z_n\}$ on $I \times [0,\infty)$, with the transition kernel

$$P((i,x), \{j\} \times [0,y)) = p_{ij} F_j(y);$$

the process $Z_n = (s_n,c_n)$ records the sequence of states visited, together with the sequence of consecutive holding times.

Suppose that P is such that there exist four subsets $\hat{A}_1$, $\hat{A}_2$, $\hat{B}_1$, $\hat{B}_2$ on I such that (6.1) is satisfied, with the chain $\{s_n\}$

playing the role of $\{X_n\}$, and $\hat{A}_1$, $\hat{A}_2$, $\hat{B}_1$, $\hat{B}_2$ replacing A_1, A_2, B_1, B_2. Then, the chain $Z_n = (s_n, c_n)$ itself satisfies (6.1), using the sets $A_i = \hat{A}_i \times [0,\infty)$, $B_i = \hat{B}_i \times [0,\infty)$.

Assuming that one requires the time required for $\{\theta_t\}$ to make a passage from A_2 to B_2, gives that θ_t visited A_1 immediately before A_2, and that a visit to B_1 immediately precedes a visit to B_2 (see [18] for the significance of A_1, B_1), the quantity of interest is $R_n(f)$, where $f(s,c) = c$.

The interest in passage times arises from the fact that passage time quantities are often important design criteria in development of networks of queues.

We proceed by first analyzing the chain $Y_n = (X_n, X_{n+1})$.

(6.3) PROPOSITION. The chain $\{Y_n: n \geq 0\}$ is Harris recurrent, provided $\{X_n\}$ is Harris. If $\{X_n\}$ possesses an invariant probability, then $\{Y_n\}$ does also.

Proof. Let A, λ, and ϕ be as in Definition 2.2. Put $\hat{A} = E \times A$ and note that (2.2)(iii) ensures that $\hat{P}_\mu\{Y_n \in \hat{A} \text{ infinitely often}\} = 1$ for any μ on the product space E^2, where $\hat{P}$ is the kernel associated with $\{Y_n\}$ given by

$$\hat{P}((x_1,x_2), D_1 \times D_2) = I_{D_1}(x_2)\, P(x_2, D_2) \quad .$$

Of course, for $(x_1, x_2) \in \hat{A}$,

$$\hat{P}^{m+1}((x_1,x_2),\ D_1 \times D_2)$$

$$= P_{x_2}\{(X_m,\ X_{m+1}) \in D_1 \times D_2\} \geq \lambda \int_{D_1} \phi(dx_1) \int_{D_2} P(x_1,dx_2)$$

$$= \lambda\hat{\phi}(D_1 \times D_2)$$

and hence Y_n satisfies (2.2)(i) through (iii), substituting $\hat{A}$, $\hat{\phi}$, $m+1$ in place of A, ϕ, m. This suffices to prove Harris recurrence (see [12], Section 4).

Also, given that π is the unique σ-finite invariant measure of $\{X_n\}$, it is simple to verify that the measure $\hat{\pi}$ given by

$$\hat{\pi}(D_1 \times D_2) = \int_{D_1} \pi(dx_1) \int_{D_2} P(x_1,\ dx_2) \tag{6.4}$$

is both σ-finite and invariant for $\hat{P}$. A trivial consequence of the representation (6.3) is that $\hat{\pi}$ is a probability provided the same is true of π. ∎

Let $\{X_n\}$ be a Harris chain and let $D \subset \underline{E}$ satisfy $\pi(D) > 0$, where π is the invariant measure of $\{X_n\}$. Then, setting $T_0 = -1$,

$$T_{k+1} = \inf\{n > T_k\colon X_n \in D\}\ ,$$

we have that $P_x\{T_k < \infty\} = 1$ for all x (see (2.3)). Let $\{V_n\}$ be the "interblock" chain

$$\beta_n = (X_{T_n+1}, \ldots, X_{T_{n+1}})$$

taking values in $\cup_{j=0}^{\infty} E^j \times D$.

(6.5) PROPOSITION. The interblock chain $\{\beta_n\}$ is Harris recurrent, provided $\{X_n\}$ is Harris. If $\{X_n\}$ possesses an invariant probability, so does $\{\beta_n\}$.

Proof. Let π be the invariant measure of $\{X_n\}$. By (2.3), $\pi(C) > 0$, implies that

(6.6) $$P_x\{X_n \in C \text{ infinitely often}\} = 1 \qquad \text{for all } x .$$

For F a measurable subset of $\cup_{j=0}^{\infty} E^j \times D$, let

$$\pi_k^*(F) = P_\pi\{X_0 \in D, (X_1, \ldots, X_k) \in F, T_1 = k\} .$$

Assuming that $\pi_k^*(F) > 0$, choose δ sufficiently small that

$$C_F = \{x \in D: P_x\{(X_1, \ldots, X_k) \in F; T_1 = k\} > \delta\}$$

has positive π-measure. Of course, by (6.6) X_{T_n} visits C_F infinitely often P_x a.s. Let $U_0 = 0$,

$$U_{n+1} = \inf\{j \geq U_n+k: X_j \in C_F\}$$

and observe that the U_j's are a subsequence of the T_k's, so that

$$P_x\{\beta_k \notin F \text{ for all } k \geq 1\}$$

$$\leq P_x\{(X_{U_k+1}, \ldots, X_{V_{k+1}}) \notin F \text{ for } 1 \leq k \leq n\} ,$$

where $V_{k+1} = \inf\{n > U_k: X_k \in D\}$. Then, by the strong Markov property applied at time U_n, the above term can be re-expressed as

$$E_x\{P_{X_{U_n}}\{(X_1, \ldots, X_{T_1}) \notin F\}; (X_{U_{k+1}}, \ldots, X_{V_{k+1}}) \notin F , \quad 1 \leq k < n\}$$

$$\leq (1-\delta)\, P_x\{(X_{U_k+1}, \ldots, X_{V_{k+1}}) \notin F, 1 \leq k < n\} ,$$

the $1-\delta$ term resulting from the fact that $X_{U_n} \in C_F$. Repeating this argument $(n-1)$ more times shows that

$$P_x\{\beta_k \notin F \text{ for all } k \geq 1\} \leq (1-\delta)^n$$

and hence β_k visits F P_x a.s. for all x. This implies that $\{\beta_n\}$ satisfies (2.3) with π_k^* playing the role of ν, proving Harris

recurrence. Finally, given that π is an invariant probability for $\{X_n\}$, it is trivial to verify that

$$\pi^*(\cdot) = P_\pi\{X_0 \in D, (X_1, \ldots, X_{T_1}) \in \cdot\} \tag{6.7}$$

is an invariant probability for $\{\beta_n\}$. ∎

Returning now to the passage time problem, we can combine Propositions 6.3 and 6.5 to obtain the following result.

(6.8) THEOREM. Suppose that $\{X_n\}$ is a Harris chain. Then, if $\{S_k\}$ is defined through (6.1), the chain

$$\xi_k = (X_{S_k}, \ldots, X_{S_{k+1}})$$

is a Harris chain. Furthermore, if $\{X_n\}$ possesses an invariant probability, then so does $\{\xi_n\}$.

Because of the alternation condition (6.1)(iii), it follows that

$$R_n(f) = \sum_{k=S_n}^{T_n-1} f(X_k)$$

can be taken to be a measurable function of the $\{Y_n\}$ process, for any $\underline{E}$-measurable f. Since $R_n(f)$ is a functional of a Harris chain, one an immediately apply the general theory of Harris chains to

obtain sufficient conditions under which strong laws and central limit theorems hold for partial sum processes associated with $\{g(R_n(f))\}$, for real-valued Borel measurable g (typical choices of g might be $g(x) = x^k$). To be precise, it is evident that if π is a probability, then

$$\sum_{k=1}^{n} g(R_k(f))/n \to E_\pi\{g(R_1(f))\} \tag{6.9}$$

P_x a.s. for all $x \in E$, provided that the right-hand side of (6.9) is finite; the identification of the limit in (6.9) follows from (6.7). If we assume a further moment condition, then the weak regenerative embedding of Section 2, together with the CLT of Section 3, proves that there exists a constant σ^2 such that

$$n^{1/2}\Big[\sum_{k=1}^{n} g(R_k(f)) - E_\pi\, g(R_1(f))\Big] \Rightarrow N(0, \sigma^2) \ .$$

We can also obtain a simple sufficient condition for finiteness of $E_\pi\, g(R_1(f))$, in the case where $g(x) = x$.

(6.10) PROPOSITION. **If $\pi|f| < \infty$, then $E_\pi|R_1(f)| < \infty$.**

Proof. Let $h(X_{k-1}, X_k) = |f(X_{k-1})| + |f(X_k)|$ and observe that if π is the invariant measure of $Y_n = (X_{n-1}, X_1)$ then by (6.4),

$$\int h(x_1,x_2)\,\hat{\pi}(dx_1,\,dx_2) = E_\pi\, h(X_{k-1},\,X_k) = 2\pi|f| < \infty$$

so that h is $\hat{\pi}$-integrable. Now, the sequence of start times $\{S_k\}$ constitutes the consecutive hitting times of (A_1,A_2) for the Harris chain Y_n. A well-known construction of the invariant measure of a Harris chain (see [20], p. 32) shows that

$$\hat{\pi}(D) = E_{\hat{\pi}}\Big\{\sum_{j=S_k+1}^{S_{k+1}} I_D(Y_j)\Big\}/\hat{\pi}(A_1 \times A_2) \quad .$$

A classical approximation argument then proves that

$$\int h\,\hat{\pi}(dy) = E_{\hat{\pi}}\Big\{\sum_{j=S_k+1}^{S_{k+1}} h(X_{j-1},\,X_j)\Big\}/\hat{\pi}(A_1 \times A_2) \tag{6.11}$$

and hence if $\pi|f| < \infty$, the right-hand side of (6.11) is finite. But

$$E_\pi\, R_n(f) \le E_\pi\Big\{\sum_{j=S_k+1}^{S_{k+1}} |f(X_j)|\Big\} \le E_{\hat{\pi}}\Big\{\sum_{j=S_k+1}^{S_{k+1}} h(X_{j-1},\,X_j)\Big\} < \infty \quad ,$$

proving our assertion. ∎

7. A Nonlinear Storage Model

In this section, we study a discrete time storage model which is a vector version of a continuous time process studied by ÇINLAR and PINSKY (1971), and by HARRISON and RESNICK (1976). We shall prove

that the model gives rise to a Harris chain possessing an invariant probability; the argument will illustrate some techniques applicable to proving Harris recurrence in applied probability models.

The model that we shall consider concerns a finite family of interconnected reservoirs. Suppose that $S_n = (S_n(1), \ldots, S_n(\ell))$ are the storage levels at time n in each of the ℓ reservoirs. Over the interval $(n, n+1]$, the ℓ reservoirs receive additional content, from an external source, in the amount Z_{n+1}. Given the storage levels S_n at time n, a decision is made to release amounts $R_n = (R_n(1), \ldots, R_n(\ell))$ over the time interval $(n, n+1]$. Assuming that the release rule $R_n = \Gamma(S_n)$, the above formulation leads to a recursion

$$S_{n+1} = S_n + Z_{n+1} - \Gamma(S_n) \quad . \tag{7.1}$$

We now make the following assumptions on Γ:

(7.2)
- i) Γ is strictly increasing,
- ii) $s-\Gamma(s)$ is strictly increasing,
- iii) Γ is continuously differentiable,
- iv) $\Gamma(0) = 0$.

By strictly increasing, we mean that $\partial\Gamma_i/\partial s_j$ is positive, where Γ_i is the ith component of Γ.

We introduce a stochastic element into the model by assuming that $\{Z_n : n \geq 1\}$ is a sequence of non-negative independent and identically

distributed random vectors. The following requirements are imposed on the joint distribution function $F(z)$ of $\{Z_n\}$:

(7.3) i) $EZ_n < \varliminf_{\|s\| \to \infty} \Gamma(s)$,

ii) F has a continuous Lebesgue density component f which is positive on $B_\varepsilon = \{x: \Gamma(a-\varepsilon) < \Gamma(x) < \Gamma(a+\varepsilon)\}$ for some $\varepsilon > 0$, where $\Gamma(a) > EZ_n$ (inequality here is componentwise).

As previously mentioned, this model bears close resemblance to some recently studied continuous-time storage processes; it is also closely related to the one-variable discrete-time storage model of BATHER (1962).

(7.4) THEOREM. Under conditions (7.2) and (7.3), $\{S_n\}$ is a Harris chain with invariant probability.

Proof. For $\eta > 0$, let $A_\eta = \{s: \Gamma(s) \leq EZ_n+\eta\} = \{s: s \leq \Gamma^{-1}(EZ_n+\eta)\}$, which, by (7.2)(iii) and (7.3)(i), is non-empty and compact for η sufficiently small. Now, let $g(s) = \|s\|$ ($\|\cdot\|$ = sum of absolute values of components) and observe that

$$E_s\, g(S_1) = E\|s + Z_1 - \Gamma(s)\|$$

$$= \sum_{i=1}^{\ell} E\{s(i) + Z_1(i) - \Gamma_i(s)\}$$

$$= \sum_{i=1}^{\ell} s(i) + \sum_{i=1}^{\ell} (EZ_1(i) - \Gamma_i(s)) < \|s\| - \ell\eta = g(s) - \ell\eta$$

for $s \notin A_\eta$, where the second equality follows by non-negativity of $s-\Gamma(s)$. Also,

$$\sup_{s \in A_\eta} E_s\, g(S_1) = \sup_{s \in A_\eta} \|s-\Gamma(s)\| + \|EZ_n\| < \infty$$

by compactness of A_η, so that by Theorem 6.1, TWEEDIE (1976), the set A_η is uniformly positive. Hence,

$$\sup_{s \in A_\eta} E_s\, T_{A_\eta} < \infty \tag{7.5}$$

where $T_{A_\eta} = \inf\{n > 0:\ S_n \in A_\eta\}$. By (7.3)(ii), we can choose η and $\gamma < \varepsilon$ sufficiently small that $A_\eta \cap B_\gamma = \phi$; we will show that there exists n, $\delta > 0$ such that

$$\sup_{s \in A_\eta} P_s\{T_{B_\gamma} > n\} < 1-\delta \quad . \tag{7.6}$$

Let $\Phi(s) = s - \Gamma(s) + \Gamma(a-\gamma/2)$, and observe that Φ is a strictly monotone transformation, so that $\Phi^n(s)$ increases to $a - \gamma/2$ for $s \in A_\eta$. In fact, $\Phi^n(0) < \Phi^n(s)$ so that it is possible to choose an n for which

$$\Phi^n(s) > a - \gamma/2$$

uniformly over $s \in A_\eta$. Hence, for $s \in A_\eta$,

$$P_s\{T_{B_\gamma} \leq n\} \geq P_s\{S_n \in B_\gamma;\ Z_1, \ldots, Z_n \in B_{\gamma/2}\}$$

$$= P_s\{\Gamma(S_n) > \Gamma(a - \gamma/2)\ ;\ Z_1, \ldots, Z_n \in B_{\gamma/2}\}$$

$$= P_s\{\Phi^n(s) > a - \gamma/2;\ Z_1, \ldots, Z_n \in B_{\gamma/2}\}$$

$$= P_s\{Z_1, \ldots, Z_n \in B_{\gamma/2}\} = \delta > 0\ ,$$

proving (7.6). Relations (7.5) and (7.6) together show that for any $\gamma > 0$,

$$P_s\{T_{B_\gamma} < \infty\} = 1 \tag{7.7}$$

for all s. Now, use (7.3)(ii) and note that for γ sufficiently small

$$P_s\{S_1 \in C\} = P\{Z_1 + s - \Gamma(s) \in C\} \geq \lambda \int_{C \cap B_{\varepsilon/2}} dy \tag{7.8}$$

for all $s \in B_\gamma$, where λ is the lower bound on the Lebesgue density of F over B_ε. Together, (7.7) and (7.8) suffice for Harris recurrence, in light of (2.2).

The fact that the invariant measure is a probability will follow as a consequence of A_η being compact and uniformly positive. First, $\{S_n\}$ is a Harris chain with Feller kernel; i.e., $s_n \to s$

implies $P(s_n,\cdot) \Longrightarrow P(s,\cdot)$ (this is a consequence of the fact that (7.1) defines S_{n+1} as a continuous function of S_n). We may therefore apply Lemma 5.1 of [24], which shows that A_η qualifies as a positive status set. Application of Proposition 4.3 of [24] completes the proof of the theorem. ∎

The theory of Section 4 proves that under conditions (7.2) and (7.3), $\{S_n\}$ may be embedded in a regenerative environment. As a special case, consider the one variable model in which $\Gamma(s) = as$ for $0 < a < 1$. Then, $\{S_n\}$ is an autoregressive process of order 1 and

$$S_{n+1} = (1-a)\, S_n + Z_{n+1} \quad . \tag{7.9}$$

Given that $F(dz) = f(z)dz$, where f is a positive continuous density, such an autoregressive process may be regarded as regenerative when appropriately embedded. A natural question to ask is whether $\{S_n\}$ can be made regenerative without the embedding. More precisely, is the process $\{S_n\}$ <u>strongly regenerative</u> in the sense that there exist regeneration times that are measurable functions of the process $\{S_n\}$ alone? By appealing to Theorem 8.17 of [11], we see that $\{S_n\}$ can be strongly regenerative only if there exist sets C_1, C_2 of positive Lebesgue measure such that

$$P_s\{S_1 \in dy\} = f(y - (1-a)s)dy = c_s\, g(y)\, dy \tag{7.10}$$

for some function c_s, g, where (7.10) must hold for all $s \in C_1$, $y \in C_2$. Since the factorization (7.10) is not generally valid, we have proven existence of a chain that can be made regenerative only via an embedding.

It is also worth pointing out that some sort of density assumption for $\{Z_n\}$ is necessary, in order to obtain Harris recurrence. In Section 6 of [12], it is shown that the autoregressive process defined by (7.9) can not be embedded in a weakly regenerative environment, even in the presence of the condition $EZ_n < \infty$, if $\{Z_n\}$ has a certain atomic distribution.

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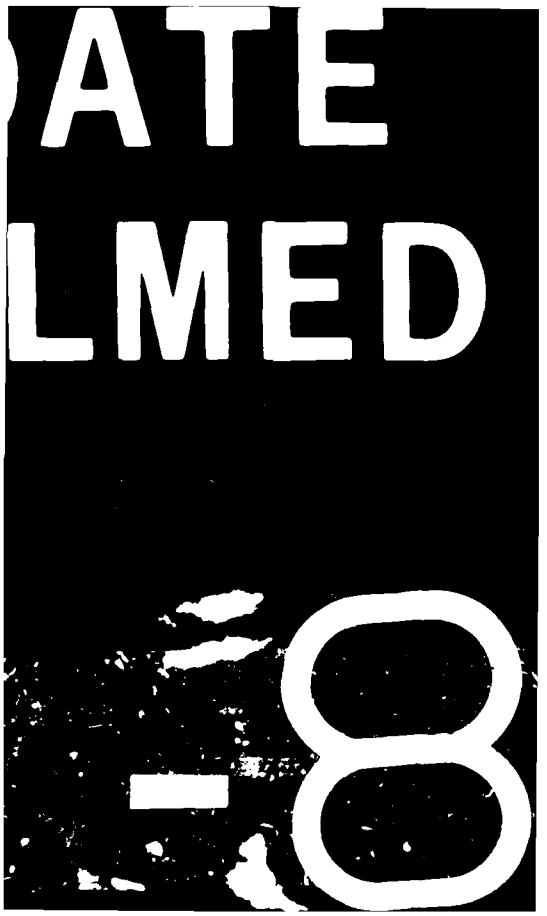
REGENERATIVE SIMULATION OF HARRIS RECURRENT MARKOV CHAINS,
by Peter W. Glynn

If the steady-state simulation problem associated with a general discrete-event simulation is well-posed, then the corresponding Markov chain is Harris recurrent. For Harris chains, it is possible to develop a simulation methodology, closely related to the regenerative method, for obtaining confidence intervals associated with estimation of steady-state parameters. The passage-time problem for Harris chains is also studied, and an estimation approach for the steady-state simulation problem is outlined. Finally, it is shown that a certain family of nonlinear storage processes is Harris recurrent--the family provides an example of a Harris chain for which the classical regenerative method is inapplicable.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